# ANALYTIC APPROACH TO THE COMPUTATION OF THE TEMPERATURE DISTRIBUTION IN MULTILAYER STRUCTURES UNDER HEATING BY CW SCANNING LASER RADIATION

A. M. Pristrem, N. I. Danilovich, and
V. A. Labunov

UDC 535.21

By using integral transform methods, an approach is developed to the solution of a problem on the temprature distribution in multilayer structures heated by cw scanning laser radiation, with phase transitions in the layers taken into account.

The application of laser radiation for materials treatment requires accurate knowledge of the distribution of the temperature fields therein. However, at this time the problem of the temperature distribution in materials heated by laser radiation and with the phase transitions and multidimensionality of the problem taken into account in general form has not been solved by either numerical or analytic methods although utilization of the former or latter methods permits solution of particular problems. Thus, heating nonlinearity, multilaminarity of the heated structures and phase transitions can easily be taken into account by numerical methods [1, 2], however, multidimensionality of the problem being solved makes difficult their application. Moreover, numerical methods are not very graphic and, therefore, are unacceptable from the viewpoint of solving the problem in general form. General and particular solutions of the above-mentioned problem have been obtained in a number of papers [5-9] by using analytic, principally integral transform [3, 4] methods, but without taking account of the phase transitions and the nonlinearity of the laser heating process. It should be noted that these two features make analytic solution most difficult, when it is a perturbation method [10] ("adiabatic" method) is developed to take account of the temperature dependence of the material thermal and optical properties, then no methods exist for taking exact account of the phase transitions to obtain an analytic solution. There are papers [11, 12] in which an attempt is made to include the phase transitions in the temperature distribution computation in particular cases in materials subjected to laser heating, however, the results obtained are only qualitative in nature.

The purpose of the present paper is the development of an analytic approach to the computation of the temperature distribution in multilayer structures under heating by cw scanning laser radiation with phase transitions taken into account.

We consider the thermal effect to be the main effect of laser action under material treatment by cw scanning laser radiation, i.e., heating of the material zone subjected to the laser action occurs to the high temperatures at which phase transitions are initiated. We assume the phase transitions abrupt, i.e., to occur at a strictly defined temperature, called the phase transition temperature. The majority of phase transitions of the I and II kind possess precisely that property. Then phase transitions can be taken into account by successively solving M heat conduction boundary value problems where the results of solving the m-th problem is taken into account in the (m + 1)-th problem. In this case the m-th boundary value problem of heat conduction is solved for the time interval in which no phase transformation is observed in the structures, i.e., separation of the whole problem into M boundary value problems is governed by the quantity of phase transformations and depends on the kind of phase transition. For phase transitions of the second kind the quantity of boundary value problems agrees with the number of phase transitions, while for phase transitions of the first kind there are twice as many boundary value problems as the number of phase transitions. An additional boundary value problem appears for each phase transition of the first kind because of the finiteness of the liberation (absorption) time of the latent heat and defines the transition zone actually existing for each phase transitions [13].

Minsk Radio Engineering Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 6, pp. 1000-1010, December, 1987. Original article submitted September 1, 1986.

 0022-0841/87/5306-1464$12.50 

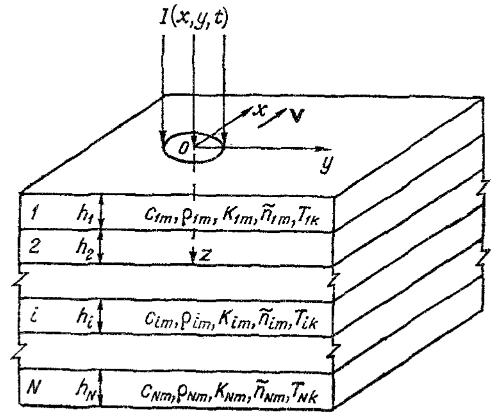


Fig. 1. Geometry of the problem of the temperature distribution in a multilayer structure under treatment by cw scanning laser radiation.

Let us consider a N-layered structure (Fig. 1) on whose surface-scanning laser radiation is normally incident along the x axis at the velocity v. Each i-th layer of this multilayered stucture of thickness $h_i$ is characterized in the m-th boundary value problem by its density $\rho_{im}$, specific heat $c_{im}$, heat conduction coefficient $K_{im}$, complex refractive index $\tilde{n}_{im} = n_{im}(1 + j\kappa_{im})$, phase transition temperatures $T_{ik}$ and latent heat of these transitions $H_{ik}$ (if the k-th phase transition in the i-th layer is of the second kind then $H_{ik} = 0$).

The algorithm to solve this problem, which we call the following point method, is the following. In the m-th boundary value problem the time $t_m$ is determined at which in the following point $\{x^0_{im};\ y^0_{im};\ z^0_{im} = z_{i-1} = \sum_{i=1}^{i-1} h_i\}$, selected such that the laser beam would be incident on it, the temperature of the k-th phase transition on the i-th layer surface would be reached and then by fixing the time $t_m$ found, the isothermal surface is found in the i-th layer that has $T_{im} = T_{ik}$. In the next (m + 1)-th problem, the thermal and optical properties of the i-th layer change at the next point and a new energetic source associated with the phase transition is also switched in. For each m-th boundary value problem the true temperature distribution in the N-layer of the structure will here be determined by the time interval $[t_{m-1};\ t_m]$ at the following point.

In the general case the temperature distribution in this problem will be described by the following system of nonlinear inhomogeneous partial differential equations:

$$c_{im}\rho_{im}\frac{\partial T_{im}}{\partial t} = \frac{\partial}{\partial r}\left(K_{im}\frac{\partial T_{im}}{\partial r}\right) + \sum_q Q_{imq}, \tag{1}$$

where i = 1, 2, ..., N; m = 1, 2, ..., M; and $Q_{imq}$ is the thermal source acting in the i-th layer for the m-th problem where the subscript q corresponds to the kind of thermal source (laser radiation to latent heat of the phase transition). The initial and boundary conditions for this problem can be represented as

$$T_{im}|_{t=t_{m-1}} = T_{i,m-1}|_{t=t_{m-1}}$$

by considering that the upper surface of the structure being treated is heat insulated and its dimensions exceed greatly the laser beam dimensions with the ideality of the thermal contact between adjacent layers of the N-layered structure and thermostatting of its lower surface taken into account, where $t_{m-1}$ (m = 1) = $t_0$ = 0 is the initial time of th eproblem under consideration and $T_{i,m-1}$(m = 1) = $T_{i0}$ = $T_0$ is the initial temperature of the multilayered structure under consideration

$$\left.\frac{\partial T_{1m}}{\partial z}\right|_{z=0} = 0, \quad T_{im}|_{z=z_i} = T_{i+1,m}|_{z=z_i},$$

$$K_{im}\frac{\partial T_{im}}{\partial z}\Big|_{z=z_i}=K_{i+1,m}\frac{\partial T_{i+1,m}}{\partial z}\Big|_{z=z_i},\quad T_{Nm}|_{z=z_N}=T_0,$$
$$T_{i1}|_{x,y\to\pm\infty}=T_0,\quad T_{im}|_{x=v(t-t_{m-1})}=T_{ik},$$
$$T_{im}|_{x=v(t'_{m-1}-t)}=T_{ik},\quad T_{im}|_{y=y_{m-1}}=T_{ik},\quad T_{im}|_{y=y'_{m-1}}=T_{ik},$$

where $y_{m-1}$, $y'_{m-1}$; $t_{m-1}$, $t'_{m-1}$ are parameters governing the boundary of the k-th phase in the i-th layer of the m-th boundary value problem determined from the solution of the preceding (m − 1)-th problem.

Application of the Kirchhoff transformations [4] to the nonlinear differential equations (1) permits writing the general solution of the problem under consideration by using integral transform methods [4, 5]. Indeed, by introducing the cited temperatures in the form of the functions

$$W_{im}(T_{im})=\int\limits_{T_{im}|_{t=t_{m-1}}}^{T_{im}}\frac{K_{im}(T'_{im})}{K_{im}|_{t=t_{m-1}}}dT'_{im},\tag{2}$$

we write the general solution for them thus (for m = 1 and m ≠ 1, respectively):

$$W_{i1}=\sum_{r=1}^{N}\int\limits_0^t d\tau\int\limits_{-\infty}^{+\infty}dx'\int\limits_{-\infty}^{+\infty}dy'\int\limits_{z_{i-1}}^{z_i}dz'G_{ir1}(x,\ y,\ z,\ t/x',\ y',\ z',\ \tau)\frac{D_{r1}}{K_{r0}}Q_{r11},$$

$$W_{im}=\sum_{r=1}^{N}\int\limits_{vt_{m-1}}^{vt'_{m-1}}dx'\int\limits_{y_{m-1}}^{y'_{m-1}}dy'\int\limits_{z_{i-1}}^{z_i}dz'G_{irm}(x,\ y,\ z,\ t/x',\ y'\ z',\ \tau)|_{\tau=t_{m-1}}\times$$
$$\times W_{i,m-1}|_{t=t_{m-1}}+\sum_{r=1}^{N}\int\limits_{t_{m-1}}^{t}d\tau\int\limits_{vt_{m-1}}^{vt'_{m-1}}dx'\int\limits_{y_{m-1}}^{y'_{m-1}}dy'\int\limits_{z_{i-1}}^{z_i}dz'\ G_{irm}(x,\ y,\ z,\ t/x',\ y'\ z',\ \tau)\frac{D_{rm}}{K_{rm}|_{t=t_{m-1}}}\sum_q Q_{rmq},$$

where $G_{irm}(x,\ y,\ z,\ t/x',\ y',\ z',\ \tau)$ is the Green's function of the m-th problem, which take the following form when the appropriate boundary conditions are taken into account (for m = 1 and m ≠ 1, respectively):

$$G_{ir1}(x,\ y,\ z,\ t/x',\ y',\ z',\ \tau)=\frac{1}{4\pi D_{i1}(t-\tau)}\exp\left\{-\left[\frac{(x-x')^2+(y-y')^2}{4D_{i1}(t-\tau)}\right]\right\}\times$$
$$\times\sum_{l=1}^{\infty}\exp\{\beta_{l1}^2(t-\tau)\}\frac{K_{r0}}{N_{l1}D_{r1}}\Psi_{il1}(z)\Psi_{rl1}(z'),$$

$$G_{irm}(x,\ y,\ z,\ t/x',\ y',\ z',\ \tau)=\frac{4}{v(t'_{m-1}-t_{m-1})(y'_{m-1}-y_{m-1})}\times$$
$$\times\sum_{s=1}^{\infty}\sum_{p=1}^{\infty}\sum_{l=1}^{\infty}\exp\{-[(\sigma_{sm}^2D_{rm}+\gamma_{pm}^2D_{rm}+\beta_{lm}^2)(t-\tau)]\}\times$$
$$\times\sin[\sigma_{sm}(x'-vt_{m-1})]\sin[\sigma_{sm}(x-vt_{m-1})]\sin[\gamma_{pm}(y'-y_{m-1})]\times$$
$$\times\sin[\gamma_{pm}(y-y_{m-1})]\frac{K_{rm}|_{t=t_{m-1}}}{N_{lm}D_{rm}}\Psi_{ilm}(z)\Psi_{rlm}(z'),$$

where $\sigma_{sm}=\pi s/v(t'_{m-1}-t_{m-1})$; $\gamma_{pm}=\pi p/(y'_{m-1}-y_{m-1})$; $D_{rm}=K_{rm}/c_{rm}\rho_{rm}$ is the thermal diffusivity of the r-th layer in the m-th problem, and $N_{lm}=\sum\limits_{r=1}^{N}[(K_{rm}|_{t=t_{m-1}})/D_{rm}]\int\limits_{z_{r-1}}^{z_r}\Psi_{rlm}^2(z')dz'$ is the norm of the m-th problem when seeking the eigenvalues $\beta_{\ell m}$ and eigenfunctions $\Psi_{r\ell m}(z)$.

It should be noted that the formalism represented for taking account of the phase transitions in the computation of temperature fields in multilayered structures heated by laser radiation can be used just for thin-layered structures for which the new phase penetrates deep into the layer during its formation in the i-th layer. The criterion of thin-laminarity of the structure will here be determined by the temperature gradients along the z axis in the layers and it can be represented in the form

$$h_i \leqslant \left(\frac{D_{im}^* d_x}{10v}\right)^{1/2}, \tag{3}$$

where $D_{im}^*$ is the value of the thermal diffusivity coefficient of the i-th layer undergoing the phase transition in the m-th problem at the phase transition temperature, and $d_x$ is the laser beam dimension along the laser radiation scanning direction.

For complete generality of the problem considered we set up the form of the thermal sources $Q_{imq}$. A thermal source in the i-th layer of the m-th problem, due to laser radiation, is represented thus

$$Q_{im1} = \frac{I[x - v(t - t_{m-1}),\ y]\, f_{im}(z)}{\Delta_{im}}, \tag{4}$$

where $I[x - v(t - t_{m-1}),\ y]$ is the space–time distribution of the laser radiation intensity over the laser beam section in the m-th problem. Here the function $f_{im}(z)$ describes the law of laser radiation energy adsorption over the depth of the i-th layer that absorbs this radiation in the m-th problem. In the case of a layer transparent to laser radiation this function is a constant. It is related to the electrical field intensity $\mathscr{E}_n$ of the incident laser radiation and the electrical field intensity in the i-th layer of the structure by the following relationship

$$f_{im}(z) = n_{im}\frac{|\mathscr{E}_{im}|^2}{|\mathscr{E}_n|^2}.$$

Using the method [14] for computing $\mathscr{E}_{im}$ in the case of a plane electromagnetic wave $\mathscr{E}_n = \mathscr{E}_0 \exp(-j\omega_0 t)$, normally incident on the surface of a N-layered structure, we obtain an analytic expression for the electrical field in the i-th layer

$$\begin{aligned}
\frac{\mathscr{E}_{im}}{\mathscr{E}_n} = & \left[\prod_{i=1}^{i}\left\{\tau_{i-1,i}\exp\left(-\frac{\alpha_{i-1,m}h_{i-1}}{2}\right)\exp\left[j\left(n_{i-1,m}\frac{2\pi}{\lambda_0}h_{i-1}+\chi_{i-1,i}\right)\right]\right\}\times\right.\\
& \times\left\{1-\eta_{i-1,i}\eta_{i-2,i-1}\exp(-\alpha_{i-1,m}h_{i-1})\exp\left[j\left(n_{i-1,m}\frac{2\pi}{\lambda_0}h_{i-1}+\right.\right.\right.\\
& \left.\left.\left.\left.+\varphi_{i-1,i}+\varphi_{i-2,i-1}\right)\right]\right\}^{-1}\right]\left\{\exp\left[-\frac{\alpha_{im}}{2}(z-z_{i-i})\right]\exp\left[j\left(n_{im}\frac{2\pi}{\lambda_0}\left(z-z_{i-1}\right)\right)\right]+\right.\\
& \left.+\eta_{i,i+1}\exp\left[-\frac{\alpha_{im}}{2}\left(h_i+z_i-z\right)\right]\exp\left[j\left(n_{im}\frac{2\pi}{\lambda_0}(2h_i+z_i-z)\right)\right]\right\}+\\
& +\left[\sum_{r\neq i}^{N}\prod_{r=1}^{N}\frac{\tau_{r-1,r}\exp\left(-\frac{\alpha_{rm}}{2}h_r\right)\exp\left[j\left(n_{rm}\frac{2\pi}{\lambda_0}2h_r+\chi_{r-1,r}\right)\right]}{1-\eta_{r,r+1}\eta_{r-1,r}\exp(-\alpha_{rm}h_r)\exp\left[j\left(n_{rm}\frac{2\pi}{\lambda_0}2h_r+\varphi_{r,r+1}+\varphi_{r-1,r}\right)\right]}\times\right.\\
& \left.\times\prod_{r\neq i}^{r}\frac{\eta_{r,r+1}\tau_{r-1,r}\exp(-\alpha_{rm}h_r)\exp\left[j\left(n_{rm}\frac{2\pi}{\lambda_0}2h_r+\varphi_{r,r+1}+\chi_{r-1,r}\right)\right]}{1-2\eta_{r,r+1}\tau_{r-1,r}\exp(-\alpha_{rm}h_r)\exp\left[j\left(n_{rm}\frac{2\pi}{\lambda_0}2h_r+\varphi_{r,r+1}+\chi_{r-1,r}\right)\right]}\right]\times\\
& \times\left\{\frac{\eta_{i-1,i}\exp\left(-\frac{\alpha_{im}h_i}{2}\right)\exp(j\varphi_{i-1,i})}{1-\eta_{i,i+1}\eta_{i-1,i}\exp(-\alpha_{im}h_i)\exp\left[j\left(n_{im}\frac{2\pi}{\lambda_0}2h_i+\varphi_{i,i+1}+\varphi_{i-1,i}\right)\right]}\right.\times
\end{aligned} \tag{5}$$

$$\times\left[\exp\left(-\frac{\alpha_{im}}{2}(z-z_{i-1})\right)\exp\left[j\left(n_{im}\frac{2\pi}{\lambda_0}(z-z_{i-1})\right)\right]+\right.$$

$$\left.\left.+\eta_{i,i+1}\exp\left(-\frac{\alpha_{im}h_i}{2}\right)\exp\left[-\frac{\alpha_{im}}{2}(z_i-z)\right]\exp\left[j\left(n_{im}\frac{2\pi}{\lambda_0}(2h_i+z_i-z)\right)\right]\right]\right\}, \qquad (5)$$

where $\tau_{i-1,i}$ and $\chi_{i-1,i}$ are the amplitude and phase of the electromagnetic wave transmission coefficients on the interface between the (i − 1)-th and i-th media, while $\eta_{i-1,i}$ and $\varphi_{i-1,i}$ are the amplitude and phase of the electromagnetic wave reflection coefficient from the interface between these media, where all these quantities are associated with the optical constants of the (i − 1)-th and i-th media by the following relationships:

$$\tau^2_{i-1,i}=\frac{4n^2_{i-1,m}(1+\varkappa^2_{i-1,m})}{(n_{i-1,m}+n_{im})^2+(n_{i-1,m}\varkappa_{i-1,m}+n_{im}\varkappa_{im})^2},$$

$$\operatorname{tg}\chi_{i-1,i}=\frac{n_{im}(\varkappa_{i-1,m}-\varkappa_{im})}{(n_{i-1,m}+n_{im})+\varkappa_{i-1,m}(n_{i-1,m}\varkappa_{i-1,m}+n_{im}\varkappa_{im})},$$

$$\eta^2_{i-1,i}=\frac{(n_{i-1,m}-n_{im})^2+(n_{i-1,m}\varkappa_{i-1,m}-n_{im}\varkappa_{im})^2}{(n_{i-1,m}+n_{im})^2+(n_{i-1,m}\varkappa_{i-1,m}+n_{im}\varkappa_{im})^2},$$

$$\operatorname{tg}\varphi_{i-1,i}=\frac{2n_{i-1,m}n_{im}(\varkappa_{i-1,m}-\varkappa_{im})}{(n^2_{i-1,m}-n^2_{im})+(n^2_{i-1,m}\varkappa^2_{i-1,m}-n^2_{im}\varkappa^2_{im})}.$$

The expression (5) consists of two components, of which the first takes account of interference in the i-th layer and the second in the remaining layers. From the viewpoint of material treatment by laser radiation, interference effects are important primarily for the first absorbing layer. If the thickness of this layer satisfies the inequality

$$h_i\geqslant\frac{4{,}6\lambda_0}{4\pi\varkappa_{im}n_{im}}, \qquad (6)$$

then interference effects therein and in successive layers can be neglected since the laser radiation intensity diminishes more than 100 times upon emerging from this layer.

The $\Delta_{im}$ in (4) is the characteristic length of the transformation of laser radiation energy into thermal energy in the i-th layer of the structure in the m-th problem, which, taking photoexcited carrier diffusion into account, is determined by the expression

$$\Delta_{im}=\frac{1}{\alpha_{im}}+\sqrt{2D_{a\,im}\tau_{\mathrm{rel}\,im}}\left(\frac{\hbar\omega_0-E_{gim}}{\hbar\omega_0}\right)+\sqrt{2D_{a\,im}\tau_{\mathrm{rec}\,im}}\,\frac{E_{gim}}{\hbar\omega_0}, \qquad (7)$$

where $\tau_{\mathrm{rel\ im}}$ is the relaxation time of the photoexcited current carrier energy determined by the electron–phonon interaction, $D_{a\ im}$ is the photoexcited current carrier diffusion coefficient, $E_{g\ im}$ and $\tau_{\mathrm{rel\ im}}$ are the forbidden bandwidth of the i-th layer material and the current carrier recombination time in this layer, respectively, that are characteristic just for semiconductor materials. The second and third components in (7) permit taking account of the physical features of the absorption process, associated with the dynamic of converting laser radiation energy into thermal energy.

The thermal source associated with the phase transition can be represented in the approximation under consideration as

$$Q_{im2}=\pm\rho_{im}H_{ik}vY_m(y)\,\delta[x-v(t-t_{m-1})],$$

where $\delta(\xi)$ is the Dirac delta-function, and $Y_m(y) = 1$ for $y \in [y_{m-i}, y'_{m-1}]$ and $Y_m(y) = 0$ for the remaining values of y. The plus and minus signs correspond to liberation or absorption of the latent heat of th ephase transition, respectively.

We consider application of the developed formalism to an example of a two-layered $Si/SiO_2$ (quartz) structure on which cw scanning laser radiation of power $P_0$ acts with a Gaussian intensity distriution over the laser beam section at a wavelength strongly absorbed in the silicon layer so that condition (6) is satisfied, and passing freely through the quartz substrate, where we consider the criterion (3) satisfied also so that the silicon layer subjected to the laser radiation melts in its whole thickness while the quartz substrate does not melt. Then the problem is separated into three boundary value problems, where there will be no thermal sources n the quartz substrate, while the thermal source in the silicon layer, due to the laser radiation, will have the form

$$Q_{1m1}=\frac{P_0(1-R_{1m})}{\pi r_x r_y \Delta_{1m}}\exp\left[-\frac{(x-vt)^2}{r_x^2}-\frac{y^2}{r_y^2}\right]\exp(-\alpha_{1m}z),$$

where $R_{im} = 1 - n_{im}\tau_{01}^2$ is the energetic coefficient of laser radiation reflection, and $r_x$ and $r_y$ are elliptical radii of the laser spot along the respective coordinate axes. Moreover, an additional heat source due to absorption of the latent heat of the transition

$$Q_{121}=\rho_{12}H_{11}vY_2(y)\,\delta(x-v(t-t_1))$$

will act in the silicon layer in the area of the phase transition in the second problem.

Let us place the tracking point at the origin. To obtain analytic expressions for the temperature in the silicon layer and in the quartz substrate we linearize the problem by using averaging over their thermal and optical properties with respect to the temperature in the temperature range from $T_0$ to $T_{me}^{Si}$. The expression to find the time $t_1$ when the temperature at the tracking pint reaches the silicon melting point is written as:

$$W_{\text{пл.}1}=W_{11}(0,\ 0,\ 0,\ t_1)=\frac{P_0(1-R_{11})}{\pi}\int_{-\infty}^{t_1}\exp\left[-\frac{(v\tau)^2}{4D_{11}(t_1-\tau)+r_x^2}\right]\times$$
$$\times[4D_{11}(t_1-\tau)+r_x^2]^{-1/2}[4D_{11}(t_1-\tau)+r_y^2]^{-\frac{1}{2}}\sum_{l=1}^{\infty}\exp[-\beta_{l1}^2(t_1-\tau)]\frac{\varphi_{l1}}{N_{l1}}\,d\tau,$$

where

$$\varphi_{l1}=\frac{D_{11}\exp(-\alpha_{11}h_1)}{\Delta_{11}(\beta_{l1}^2+\alpha_{11}^2D_{11})}\left\{\frac{\beta_{l1}}{\sqrt{D_{11}}}\operatorname{tg}\left(\frac{\beta_{l1}}{\sqrt{D_{11}}}h_1\right)-\alpha_{11}\left[1-\cos^{-1}\left(\frac{\beta_{l1}}{\sqrt{D_{11}}}h_1\right)\exp(\alpha_{11}h_1)\right]\right\},$$

$$N_{l1}=h_1\cos^{-2}\left(\frac{\beta_{l1}}{\sqrt{D_{11}}}h_1\right)\frac{K_{11}|_{t=0}}{2D_{11}}+h_2\sin^{-2}\left(\frac{\beta_{l1}}{\sqrt{D_{21}}}h_2\right)\frac{K_{21}|_{t=0}}{2D_{21}},$$

and $\beta_c$ is found from the solution of the transcendental equation

$$\frac{K_{11}|_{t=0}}{\sqrt{D_{11}}}\operatorname{tg}\left(\frac{\beta_{l1}}{\sqrt{D_{11}}}h_1\right)=\frac{K_{21}|_{t=0}}{\sqrt{D_{21}}}\operatorname{ctg}\left(\frac{\beta_{l1}}{\sqrt{D_{21}}}h_2\right).$$

The reduced temperature $W_{im}$ is easily related to the real temperature $T_{im}$ by using (2). Thus, for silicon [15]

$$K_{11}(T_{11})=\frac{K_c}{T_{11}-T_c},$$

where $K_c$ and $T_c$ are empirical constants so that

$$W_{11}=(T_0-T_c)\ln\left(\frac{T_{11}-T_c}{T_0-T_c}\right). \tag{8}$$

After this the equation to find the isothermal surface can be written

$$W_{\text{пл.}1}=W_{11}(x,\ y,\ z,\ t_1)=\frac{P_0(1-R_{11})}{\pi}\int_{-\infty}^{t_1}\frac{\exp\left[-\dfrac{(x-v\tau)^2}{4D_{11}(t_1-\tau)+r_x^2}\right]}{[4D_{11}(t_1-\tau)+r_x^2)^{1/2}}\times$$

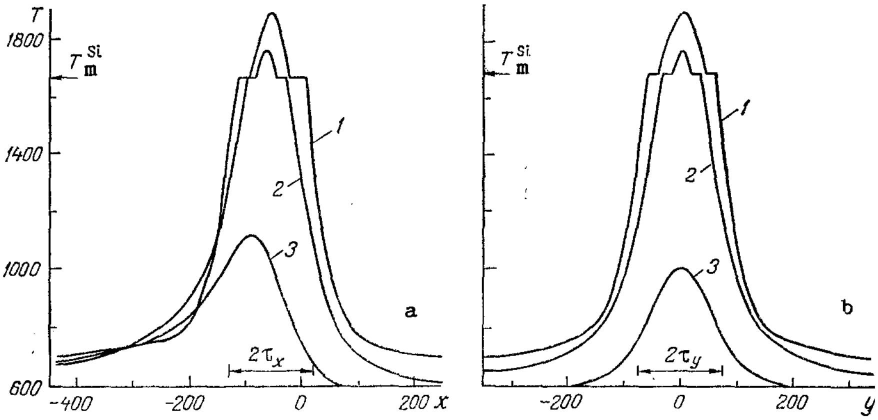


Fig. 2. Stationary temperature distribution profiles relative to the moving laser beam on the two-layered $Si/SiO_2$ structure surface in the laser radiation scanning direction (a) and perpendicular to it (b) for three scanning speeds: 0.01 (1); 0.1 (2) and 1 m/sec (3); T, K; x, y, μm. Segments are $2\tau_x$ and $2\tau_y$.

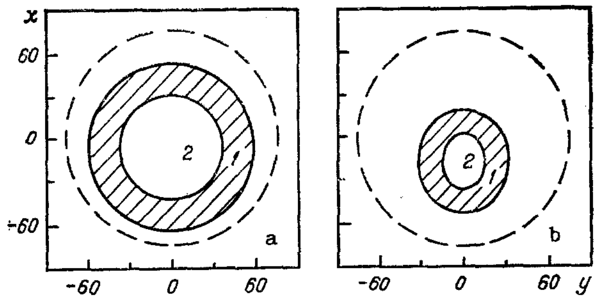


Fig. 3. Domains of the transition zone 1 and melting zone 2 on the two-layer $Si/SiO_2$ structure surface for two scanning rates 0.01 (a) and 0.1 m/sec (b) computed in the stationary mode. Dashes show boundary of laser radiation spot.

$$\times \frac{\exp\left[-\frac{y^2}{4D_{11}(t_1-\tau)+r_y^2}\right]}{[4D_{11}(t_1-\tau)+r_y^2]^{1/2}} \sum_{l=1}^{\infty} \exp[-\beta_{l1}^2(t_1-\tau)] \frac{\varphi_{l1}}{N_{l1}} \cos\left(\frac{\beta_{l1}}{\sqrt{D_{11}}} z\right) d\tau,$$

outside of which we have the true temperature distribution.

Now, solving the second problem we determine the time $t_2$ when total melting of the silicon layer occurs at the tracking point. Setting $R_{12} \approx R_{11}$, $D_{12} \approx D_{11}$, $c_{12} \approx c_{11}$, $\rho_{12} \approx \rho_{11}$, $\alpha_{12} \approx \alpha_{11}$, we obtain

$$W_{\text{пл.1}} = W_{12}(0,\ 0,\ 0,\ t_2) = W_{11}(0,\ 0,\ 0,\ t_2) - \frac{H_{11}\rho_{12}v}{4\sqrt{\pi}} \times$$
$$\times \sum_{l=1}^{\infty} \int_{-\infty}^{t_2} \frac{\exp[-\beta_{l2}^2(t_2-\tau)] \operatorname{tg}\left(\frac{\beta_{l2}}{\sqrt{D_{12}}} h_1\right)}{\beta_{l2}(t_2-\tau)^{1/2} N_{l2} \cos\left(\frac{\beta_{l2}}{\sqrt{D_{12}}} h_1\right)} \exp\left[-\frac{(v\tau)^2}{4D_{12}(t_2-\tau)}\right] \times$$
$$\times \left\{\operatorname{erf}\left(\frac{y_1'}{\sqrt{4D_{12}(t_2-\tau)}}\right) + \operatorname{erf}\left(\frac{y_1}{\sqrt{4D_{12}(t_2-\tau)}}\right)\right\} d\tau,$$

where $\beta_{\ell 2} \approx \beta_{\ell 1}$, $N_{\ell 2} \approx N_{\ell 1}$.

Then fixing $t_2$, we determine the surface bounding the domain of liquid silicon

$$W_{m+1} = W_{12}(x, y, z, t_2) = W_{11}(x, y, z, t_2) - \frac{H_{11}\rho_{12}v}{4\sqrt{\pi}} \times \sum_{l=1}^{\infty} \int_{-\infty}^{t_2} \frac{\exp[-\beta_{l2}^2 (t_2-\tau)] \operatorname{tg}\left(\frac{\beta_{l2}}{\sqrt{D_{12}}} h_1\right)}{\beta_{l2}(t_2-\tau)^{1/2} N_{l2} \cos\left(\frac{\beta_{l2}}{\sqrt{D_{12}}} h_1\right)} \cos\left(\frac{\beta_{l2}}{\sqrt{D_{12}}} z\right)$$

$$\times \exp\left[-\frac{(x-v\tau)^2}{4D_{12}(t_2-\tau)}\right]\left\{\operatorname{erf}\left(\frac{y_1'-y}{\sqrt{4D_{12}(t_2-\tau)}}\right) + \operatorname{erf}\left(\frac{y-y_1}{\sqrt{4D_{12}(t_2-\tau)}}\right)\right\} d\tau.$$

It should be noted that $W_{12}$ is also related to $T_{12}$ by the relationship (8), however $T_{12}$ is not the true temperature since the true temperature remains constant and equal to the silicon melting point from the time $t_1$ to the time $t_2$ at the tracking point.

Solving the third problem we obtain the real temperature distribution in the liquid silicon domain.

Represented in Fig. 2 are pure temperature distribution profiles on the structure surface, computed by using an electronic computer, for and perpendicular to the laser radiation scanning direction, respectively, for three scanning rates where the computations are presented in a coordinate system coupled to the moving laser beam in the stationary mode when the temperature distribution relative to the laser beam is invariant. It is assumed here that: $P_0 = 6$ W; $r_x = r_y = 7.5 \cdot 10^{-5}$ m; $\lambda_0 = 1.06 \cdot 10^{-6}$ m; $T_0 = 600$ K; $R_{11} = 0.4$; $T_c = 99$ K; $K_c = 2.99 \cdot 10^4$ W/m; $c_{11} = 0.8 \cdot 10^3$ J/(kg·K); $\rho_{11} = 2.3 \cdot 10^3$ kg/m³; $T_m^{Si} = 1683$ K; $h_1 = 1 \cdot 10^{-6}$ m; $D_{11} = 2 \cdot 10^{-5}$ m²/sec; $\alpha_{11} = 10^6$ m⁻¹; $K_{21} = 2$ W/(m·K); $D_{21} = 10^{-6}$ m²/sec; $c_{21} = 0.88 \cdot 10^3$ J/(kg·K); $\rho_{21} = 2.2 \cdot 10^3$ kg/m³; $h_2 = 4 \cdot 10^{-4}$ m, which correspond to experiment conditions [16].

Represented in Fig. 3 are the domains of the laser spot, the transition zone, and the melt zone on the structure surface, computed in the stationary melting mode, for two laser radiation scanning rates.

Analysis of the results obtained shows that the width of the total melt domain and transition zone agrees to within 10% accuracy with the experiment results [16], which indicates the method is correct.

Therefore, the analytic method developed to solve temperature distribution problems in multilayered structures heated by cw scanning laser radiation permits taking account of phase transformations in the layers and can be used to model laser material treatment processes. It should be noted that since the problem being solved in general form is not stationary, the formalism elucidated is suitable also for temperature distribution computations in multilayered structures treated by laser radiation pulses. Moreover, the sphere of problems solvable by this method is extended by replacing the laser radiation by other energetic radiation (electronic, ionic, photonic, etc.). To do this, (4) must be replaced by the appropriate energy absorption law for these radiations over the bulk of the i-th layer.

## NOTATION

$n_{im}$, refractive index of the i-th layer of the multilayered structure in the m-th boundary-value problem; $\kappa_{im}$, extinction coefficient of the i-th layer of the structure in the m-th problem; $\lambda_0$, laser radiation wavelength; $\omega_0$, electromagnetic field frequency of the laser radiation; $\hbar$, Planck's constant; $j = \sqrt{-1}$, imaginary unit; and $\alpha_{im} = 4\pi/\lambda_0 \, \kappa_{im} n_{im}$, laser radiation absorption coefficient in the i-th layer of the structure.

## LITERATURE CITED


1. P. Baeri, S. U. Campisano, G. Foti, and E. Rimini, J. Appl. Phys., 50, No. 2, 788-797 (1979).
2. R. F. Wood and G. E. Giles, Phys. Rev., 23, No. 6, 2923-2942 (1981).
3. A. V. Lykov, Theory of Heat Conduction [in Russian], Moscow (1967).
4. M. N. Ozisik, Heat Conduction (1980).
5. U. C. Peak and A. Kestenbaum, J. Appl. Phys., 44, No. 5, 2260-2268 (1973).
6. M. Laz, J. Appl. Phys., 48, No. 9, 3919-3924 (1977).

7. Y. I. Nissim, A. Lietoila, R. B. Gold, and J. F. Gibbons, J. Appl. Phys., 50, No. 1, 274-279 (1979).
8. M. L. Bürgener and R. E. Reedy, J. Appl. Phys., 53, No. 6, 4357-4363 (1982).
9. I. D. Calden and R. Sue, J. Appl. Phys., 53, No. 11, 7545-7550 (1982).
10. D. M. Kim, R. R. Shah, and D. L. Cristhwait, J. Appl. Phys., 51, No. 6, 3121-3125 (1980).
11. J. E. Mody and R. H. Hendel, J. Appl. Phys., 53, No. 6, 4364-4371 (1982).
12. S. A. Kokorowski, G. L. Olsen and L. D. Hess, Laser and Electron Beam Solid Interaction and Materials Processing, N. Y. (1981), pp. 139-145.
13. W. H. Hawkins and D. K. Biegelsen, Appl. Phys. Lett., 42, No. 4, 358-360 (1983).
14. M. Born and E. Wolf, Principles of Optics, Pergamon (1970).
15. C. Y. Ho, R. W. Powell, and P. E. Liley, J. Phys. Chem, 3, Suppl. 1, 558-590 (1974).
16. M. Kimura and Y. Egami, Appl. Phys. Lett., 44, No. 4, 420-422 (1984).

# SOME FORMULAS OF OPERATIONAL CALCULUS FOR STEP FUNCTIONS GENERATED BY SPECIAL FUNCTIONS

Yu. A. Brychkov and A. P. Prudnikov

UDC 517.942.82:536.24

One-dimensional Laplace transforms of some step functions are presented.

Formulas of operational calculus for step functions play an important role in discrete analysis, in particular, in the solution of various equations [1, 2]. In the present paper, which is a continuation of [3], we present a table of new operational relationships for step functions that are special functions of the function $[t]$, the integer part of t. The relationships are presented in two columns: the function $f([t])$ in the left hand column and its Laplace transform F(p) in the right hand column, where

$$F(p)=\int_0^\infty f([t])\exp(-pt)\,dt=\frac{1-e^{-p}}{p}\sum_{k=0}^{\infty} f(k)\,e^{-kp}.$$

Here $f([t]) = f(k)$ for $k \leq t < k + 1$, $k = 0, 1, 2, \ldots$; $\operatorname{Re} p > 0$, unless the contrary is indicated. The notation employed is that commonly used in the mathematical literature (see, for example, [4-7]).

TABLE 1. Laplace Transforms of Some Step Functions

| No. | $f([t])$ | $F(p)$ |
|---|---|---|
| 1 | $x^{[t]}\Gamma([t],\ a)$ | $\frac{1-e^{-p}}{p}\left[\exp(xe^{-p})\operatorname{Ei}\left(-a+\frac{e^p}{x}\right)-\operatorname{Ei}(-a)\right]$, $\lvert axe^{-p}\rvert<1$ |
| 2 | $\frac{x^{[t]}}{[t]!}\Gamma([t],\ a)$ | $\frac{1-e^{-p}}{p}\operatorname{Ei}(axe^{-p}-a),\quad a(1-xe^{-p})>0$ |
| 3 | $x^{[t]+1}\zeta([t]+2)$ | $\frac{1-e^{p}}{p}[\mathbf{C}+\psi(1-xe^{-p})],\quad \lvert xe^{-p}\rvert<1$ |
| 4 | $x^{2[t]}\zeta(2[t])$ | $\frac{\pi x}{2}\,\frac{e^{-3p/2}-e^{-p/2}}{p}\operatorname{ctg}(\pi xe^{-p/2}),\quad \lvert xe^{-p/2}\rvert<1$ |

Computing Center, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 6, pp. 1010-1014, December, 1987. Original article submitted November 11, 1986.

 0022-0841/87/5306-1472$12.50 